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Instability of a viscous fluid of variable density in a magnetic field

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The instability to small two-dimensional disturbances of an electrically conducting fluid of variable density is investigated. The viscous fluid is bounded between two vertical parallel planes normal to which a magnetic field of constant intensity is applied. Significant parameters upon which the behaviour of the Rayleigh number at neutral stability depends are the Hartmann number M and the wave-number α which is associated with a periodic disturbance with periodicity in the unbounded horizontal direction.

The solution may be sought by considering basic disturbances which are either symmetric or antisymmetric about the median plane parallel to the boundary planes. It is found that for a given magnetic field strength the critical Rayleigh number governing stability is associated with an antisymmetric disturbance of zero wave-number. The least stable symmetric disturbance which arises when the wave-number is zero is less easily excited. This trend is seen again in the purely hydrodynamic case ($M = 0$) where, corresponding to a finite wave-number value, the more unstable mode at neutral stability is found to be an antisymmetric one.

The most unstable situation occurs when both the Hartmann number and the wave number are zero. In this case the result of Wooding (1960) that the minimum critical Rayleigh number is zero and is associated with a symmetric disturbance is reobtained.

1. Introduction

Exact solutions of the instability to small two-dimensional disturbances of a viscous variable density fluid bounded by parallel vertical planes have been obtained by Ostrach (1955) and Yih (1959) the last mentioned citing the previous work of Taylor (1954) in this connexion. Both authors have considered the bounding planes to be insulated with the fluid subjected to a negative upward temperature gradient. Yih has shown that, if disturbances which are periodic in the vertical direction with wave number γ are assumed, then the most unstable modes are associated with a zero wave-number. Further the trend of his results indicated that a disturbance exhibiting antisymmetry about the vertical median plane would be more readily excited for a stated wave-number than one having a similar property of symmetry. Corresponding to an antisymmetric mode of zero wave-number he therefore obtained the value $R = 31{\cdot}29$ for the overall critical Rayleigh number at neutral stability. For the most unstable symmetric disturbance Yih and Ostrach have both given the value $R = 237 \cdot 6$. However, Wooding (1960), who reinvestigated the problem in connexion with the stability of a variable density liquid in a porous medium, found that a two-dimensional symmetric disturbance with variation confined to any horizontal plane (i.e. with $\gamma = 0$) and periodic with wave-number α parallel to the bounding planes furnished a more unstable mode than either of those mentioned above. In case α is small and significant only to a first power he obtained a zero value for the critical Rayleigh number in association with a vertical velocity component of plane parabolic profile and of magnitude proportional to a first power of α. A physical interpretation of this result which is anomalous to the general trend of critical Rayleigh number values indicated by the results of Yih has been offered by Wooding.

The present investigation is concerned with the stability of a column of viscous electrically conducting fluid bounded by two parallel vertical planes normal to which a uniform magnetic field is applied. The planes are thermally insulated and the fluid is heated from below. The relevant equations of motion and thermal energy transport use the standard Boussinesq approximation provided that the temperature difference between any two points of the fluid is moderate in the sense that $\phi(T_1 - T_2) \ll 1$ where ϕ is the coefficient of thermal expansion. Density variations are therefore small and the only effect of these which may not be ignored is on the body force per unit volume due to gravity since this is the sole cause of instability. For onset of convection the linearized governing equations are reduced to their simplest two-dimensional form by assuming that the velocity field comprises only one component which is in the upward direction.

The principle of exchange of stabilities, i.e. that instability sets in as time-independent convection, is shown to be valid for the problem which reduces to the solution of a transcendental equation of simple form. Of particular interest is the effect of the magnetic field on the most unstable mode discovered by Wooding. In fact there is no evidence of this mode when the fluid is electrically conducting.

2. Basic equations

In a standard notation employing emu and cgs units the basic equations are those governing the velocity and temperature, viz.

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\,.\,\nabla\mathbf{v}\right) = -\nabla p + \nu\nabla\,.\,(\rho\nabla\mathbf{v}) + \rho\mathbf{g} + \mathbf{F}, \tag{2.1}$$

$$\frac{\partial \rho}{\partial t} + \nabla\,.\,(\rho\mathbf{v}) = 0, \tag{2.2}$$

$$\rho\left(\frac{\partial T}{\partial t} + \mathbf{v}\,.\,\nabla T\right) = K\nabla\,.\,(\rho\nabla T), \tag{2.3}$$

together with the electromagnetic equations

$$\nabla\,.\,\mathbf{H} = 0, \quad \nabla\times\mathbf{H} = 4\pi\mathbf{J}, \quad \nabla\times\mathbf{E} = -\mu\partial\mathbf{H}/\partial t, \tag{2.4, 5, 6}$$

and Ohm's law for the moving field

$$\mathbf{J} = \sigma(\mathbf{E} + \mu\mathbf{v}\times\mathbf{H}). \tag{2.7}$$

Viscous and electrical dissipations have been neglected in (2.3) and $\mathbf{F}$ in (2.1) represents the induced magnetic force per unit volume.

Cartesian co-ordinates (x, y, z) are used with z measured in the vertical direction. The vertical bounding planes are parallel to the (y, z)-plane and intersect the x-axis at $x = \mp d$. Normal to the planes, i.e. in the direction of the x-axis a uniform magnetic field of indensity H_0 is applied.

An equation of state valid for small temperature differences can be written

$$\rho = \rho_0[1-\phi(T-T_0)]$$

in which ϕ is the coefficient of thermal expansion and ρ_0 and T_0 are respectively the fluid density and temperature at the chosen origin. The coefficient ϕ is small—for mercury at room temperature (20 °C) it is $1{\cdot}82\times 10^{-4}$—and in general may be considered not greater than 10^{-3} so that for moderate temperature differences the variation in the density ρ is everywhere slight. In the primary static state one has

$$T = T_0+\beta z, \quad \rho = \rho_0(1-\phi\beta z), \quad dp/dz = -g\rho_0(1-\phi\beta z), \tag{2.8}$$

where β is the (negative) upward temperature gradient. With convection present the perturbations from the initial state described by the equations (2.8) are suitably denoted by $\mathbf{v}$, T', p', ρ', $\mathbf{H}$ and these are small quantities significant only to a first order. On the Boussinesq approximation that density variations are felt only in relation to the gravitational body force, equations (2.1) to (2.3) using (2.8) can be expressed in the linearized forms

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho_0}\nabla p' + \nu\nabla^2\mathbf{v} + \frac{1}{\rho_0}\mathbf{F} + (0, 0, \phi g T'), \tag{2.9}$$

$$\nabla . \mathbf{v} = 0, \tag{2.10}$$

$$\frac{\partial T'}{\partial t} = K\nabla^2 T' - \beta w, \tag{2.11}$$

where w appearing in (2.11) is the component of $\mathbf{v}$ in the upward direction.

Consistent with the governing equations and boundary conditions we assume that the velocity field has only one non-zero component w, and that H_z is the only component of the induced magnetic field. From (2.10) and (2.4) therefore the total velocity and magnetic fields are given by

$$\left.\begin{aligned} u = v = 0, \quad w = w(x, y), \\ H_x = H_0, \quad H_y = 0, \quad H_z = H_z(x, y). \end{aligned}\right\} \tag{2.12}$$

The equation (2.5) then affords the components of the current density

$$J_x = \frac{1}{4\pi}\frac{\partial H_z}{\partial y}, \quad J_y = -\frac{1}{4\pi}\frac{\partial H_z}{\partial x}, \quad J_z = 0, \tag{2.13}$$

whence the components of the magnetic force $\mathbf{F} = \mu\mathbf{J}\times\mathbf{H}$ in (2.9) are found to be

$$F_x = -\frac{\mu}{4\pi}H_z\frac{\partial H_z}{\partial x}, \quad F_y = -\frac{\mu}{4\pi}H_z\frac{\partial H_z}{\partial y}, \quad F_z = \frac{\mu}{4\pi}H_0\frac{\partial H_z}{\partial x}. \tag{2.14}$$

The first two equations of (2.14) show that F_x and F_y are quadratic in H_z and so vanish to the first order. It is now possible to write down the governing equations in their final forms. If $\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$, the equations (2.9) and (2.11) for the velocity and temperature distributions become respectively

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + \phi g T' + \nu \nabla^2 w + \frac{\mu}{4\pi\rho_0} H_0 \frac{\partial H_z}{\partial x}, \tag{2.15}$$

and

$$\frac{\partial T'}{\partial t} = K\nabla^2 T' - \beta w. \tag{2.16}$$

The linearized equation for the variable magnetic field component H_z is

$$\frac{\partial H_z}{\partial t} = \frac{1}{4\pi\sigma\mu}\nabla^2 H_z + H_0 \frac{\partial w}{\partial x}, \tag{2.17}$$

which is derived from the equations (2.4) to (2.7) together with (2.10). In addition to (2.15) the equation (2.9) gives the further equations $0 = \rho_0^{-1}(\partial p'/\partial x)$ and $0 = \rho_0^{-1}(\partial p'/\partial y)$ from which it is clear that $\partial p'/\partial z$ in (2.15) although functional in time is independent of the space variables. Finally, since there is no heat transmission through the vertical planes at which a tangential discontinuity of the magnetic field is inadmissible, the boundary conditions are

$$w = 0, \quad H_z = 0, \quad \partial T'/\partial x = 0 \quad \text{at} \quad x = \pm d. \tag{2.18}$$

The equations (2.15) to (2.18) may be rendered dimensionless by the transformations

$$\bar{t} = \nu t/d^2, \quad \bar{x} = x/d, \quad \bar{y} = y/d, \quad \bar{T} = T'/\beta d, \quad \bar{w} = wd/K \quad \text{and} \quad \bar{H}_z = H_z/H_0.$$

Substituting these and immediately dropping bars one obtains

$$\frac{\partial w}{\partial t} = -\frac{d^2}{\rho_0 \nu K}\frac{\partial p'}{\partial z} - RT + \nabla^2 w + \eta M^2 \frac{\partial H_z}{\partial x}, \tag{2.19}$$

$$Pr\frac{\partial T}{\partial t} = \nabla^2 T - w, \tag{2.20}$$

and

$$Pr\frac{\partial H_z}{\partial t} = \eta\nabla^2 H_z + \frac{\partial w}{\partial x} \tag{2.21}$$

in place of (2.15), (2.16) and (2.17). Here $Pr = \nu/K$ is the Prandtl number, $R = -\phi g \beta d^4/K\nu$ the Rayleigh number, $M^2 = \sigma\mu^2 H_0^2 d^2/\rho_0 \nu$ the Hartmann number squared and $\eta = (4\pi\mu\sigma K)^{-1}$ is the ratio of magnetic diffusivity to thermal diffusivity of the fluid.

These equations permit the separable solutions

$$w = V(x,y)\,e^{\lambda t}, \quad T = \theta(x,y)\,e^{\lambda t}, \quad H_z = H(x,y)\,e^{\lambda t}, \quad \partial p'/\partial z = P e^{\lambda t}, \tag{2.22}$$

in which λ is assumed complex and P is a constant. The separated equations (2.19) to (2.21) then become

$$\lambda V = -\frac{Pd^2}{\rho_0 \nu K} - R\theta + \nabla^2 V + \eta M^2 \frac{\partial H}{\partial x}, \tag{2.23}$$

$$Pr\lambda\theta = \nabla^2\theta - V, \tag{2.24}$$

$$Pr\lambda H = \eta\,\nabla^2 H + \frac{\partial V}{\partial x}, \tag{2.25}$$

with boundary conditions from (2.18)

$$V = 0, \quad H = 0, \quad \partial\theta/\partial x = 0 \quad \text{at} \quad x = \pm 1. \tag{2.26}$$

3. Solution

Provided the principle of exchange of stabilities is valid, the imaginary as well as the real part of λ may be equated to zero at neutral stability. It is convenient to defer until later a proof of the principle in the present instance and for the time being to assume that it holds. The elimination of θ and H between the equations (2.23), (2.24) and (2.25) then results in the equation

$$\left(\nabla^4 - M^2 \frac{\partial^2}{\partial x^2}\right) V = RV, \tag{3.1}$$

and if a disturbance of the form

$$V(x,y) = f(x) \cos \alpha y \tag{3.2}$$

is assumed, (3.1) in turn leads to

$$[(D^2-\alpha^2)^2 - M^2D^2]f = Rf, \tag{3.3}$$

in which $D \equiv d/dx$. In virtue of the conditions (2.26), one has at the boundaries

$$\nabla^2(\partial V/\partial x) = -\eta M^2(\partial^2 H/\partial x^2), \tag{3.4}$$

from (2.23) and

$$\eta(\partial^2 H/\partial x^2) = -\partial V/\partial x \tag{3.5}$$

from (2.25). Combining (3.4) and (3.5) gives

$$(\nabla^2 - M^2)\, \partial V/\partial x = 0. \tag{3.6}$$

The boundary conditions on $f(x)$ obtained from (2.26) and (3.6) are therefore

$$f(x) = 0, \quad (D^2-\alpha^2-M^2)\,Df = 0 \quad \text{at} \quad x = \pm 1. \tag{3.7}$$

The equation (3.3) has the solution

$$f(x) = \sum_i B_i \cosh \epsilon_i x + \sum_i C_i \sinh \epsilon_i x \quad (i = 1, 2), \tag{3.8}$$

where $\epsilon_i^2 - \alpha^2 = \omega_i$ $(i = 1,2)$ are the roots of the quadratic equation

$$\omega^2 - M^2\omega - M^2\alpha^2 - R = 0.$$

Thus, for the ϵ_i, we have

$$\left.\begin{aligned} 2\epsilon_1^2 &= M^2 + (M^4 + 4M^2\alpha^2 + 4R)^{\frac{1}{2}} + 2\alpha^2, \\ 2\epsilon_2^2 &= M^2 - (M^4 + 4M^2\alpha^2 + 4R)^{\frac{1}{2}} + 2\alpha^2, \end{aligned}\right\} \tag{3.9}$$

and it is evident from these expressions that

$$\left(\frac{\partial \epsilon_i}{\partial \alpha}\right)_{\alpha=0} = 0, \quad \left(\frac{\partial \epsilon_i}{\partial M}\right)_{M=0} = 0 \quad (i = 1, 2). \tag{3.10}$$

In view of the linear homogeneous character of (3.3) the general solution (3.8) may be separated into its even and odd components and these investigated individually. They will correspond respectively to velocity perturbations which are symmetric and antisymmetric about the median plane $x = 0$.

If the motion is antisymmetric the boundary conditions (3.7) afford the secular equation

$$F(R,M,\alpha) \equiv \begin{vmatrix} \sinh\epsilon_1 & \sinh\epsilon_2 \\ (\epsilon_1^2-\alpha^2-M^2)\,\epsilon_1\cosh\epsilon_1 & (\epsilon_2^2-\alpha^2-M^2)\,\epsilon_2\cosh\epsilon_2 \end{vmatrix} = 0, \quad (3.11)$$

from which R values may be obtained for prescribed M and α at neutral stability. This relationship (3.11) provides a real equation since the determinant after expansion has terms which are of odd order in ϵ_2 which is pure imaginary when R is its dominant parameter. It is clear that if the determinant in (3.11) is differentiated with respect to M the result is the sum of two determinants each having a row (column) of elements containing one or other of the factors

$$\partial\epsilon_i/\partial M \quad (i = 1,2).$$

A similar conclusion holds if the differentiation is performed with respect to α only this time one or other of the factors $\partial\epsilon_i/\partial\alpha\,(i = 1,2)$ is contained by each element of a row (column) of the derived determinants. Differentiating the equation (3.11) one obtains the results

$$\frac{\partial R}{\partial M} = -\frac{\partial F}{\partial M}\bigg/\frac{\partial F}{\partial R} \quad \text{and} \quad \frac{\partial R}{\partial \alpha} = -\frac{\partial F}{\partial \alpha}\bigg/\frac{\partial F}{\partial R},$$

whence it follows immediately using (3.10) that

$$(\partial R/\partial M)_{M=0} = 0 \quad \text{and} \quad (\partial R/\partial\alpha)_{\alpha=0} = 0.$$

That these provide criteria for minimizing R seems assured since the stability of the fluid would be increased by magnetic and viscous stress if M and α are increased from zero.

In regard to symmetric motion, the secular equation

$$\begin{vmatrix} \cosh\epsilon_1 & \cosh\epsilon_2 \\ (\epsilon_1^2-\alpha^2-M^2)\,\epsilon_1\sinh\epsilon_1 & (\epsilon_2^2-\alpha^2-M^2)\,\epsilon_2\sinh\epsilon_2 \end{vmatrix} = 0 \qquad (3.12)$$

is obtained from the boundary conditions (3.7). The above conclusions can be repeated here in relation to (3.12) and are therefore universally true.

The task of finding critical Rayleigh numbers is now very much simpler. One can say that for a disturbance of given wave-number α, the most critical situation arises when the fluid is electrically non-conducting. Alternatively one can conclude that for a given magnetic field strength the most critical Rayleigh number corresponds to a zero wave-number.

Of particular interest in case of either symmetric or antisymmetric motion are (i) the variation of critical Rayleigh number with wave-number when the Hartmann number M is zero and (ii) the behaviour of the critical Rayleigh number with Hartmann number when the fluid is electrically conducting. In view of what has been said above α is taken to be zero if $M \neq 0$ in the latter case.

Case (i), $M = 0$

Here the evaluation of the secular equations (3.11) and (3.12) lead to the transcendental equations

$$\frac{\tanh\,(R^{\frac{1}{2}}-\alpha^2)^{\frac{1}{2}}}{(R^{\frac{1}{2}}-\alpha^2)^{\frac{1}{2}}} + \frac{\tanh\,(R^{\frac{1}{2}}+\alpha^2)^{\frac{1}{2}}}{(R^{\frac{1}{2}}+\alpha^2)^{\frac{1}{2}}} = 0 \qquad (3.13)$$

for antisymmetric convection, and

$$(R^{\frac{1}{2}}+\alpha^2)^{\frac{1}{2}}\tanh(R^{\frac{1}{2}}+\alpha^2)^{\frac{1}{2}}-(R^{\frac{1}{2}}-\alpha^2)^{\frac{1}{2}}\tan(R^{\frac{1}{2}}-\alpha^2)^{\frac{1}{2}} = 0 \tag{3.14}$$

for symmetric convection.

Corresponding to $\alpha = 0$ it is then possible to calculate the most critical Rayleigh number appropriate to either mode of convection from the respective equations

$$R^{-\frac{1}{4}}(\tanh R^{\frac{1}{4}}+\tan R^{\frac{1}{4}}) = 0, \tag{3.15}$$

and

$$R^{\frac{1}{4}}(\tanh R^{\frac{1}{4}}-\tan R^{\frac{1}{4}}) = 0. \tag{3.16}$$

The least values of R satisfying these are respectively $R = 31{\cdot}285$ and $R = 0$.

α	R: Antisymmetric convection	R: Symmetric convection
0	31·285	0
0·5	35·216	247·47
1·0	47·832	277·43
2·5	170·10	519·01
3·0	257·40	668·19

TABLE 1. Critical values of R for given α ($M = 0$).

Previously Yih (1959) had obtained the equations

$$\tanh R^{\frac{1}{4}}+\tan R^{\frac{1}{4}} = 0 \tag{3.17}$$

and

$$\tanh R^{\frac{1}{4}}-\tan R^{\frac{1}{4}} = 0 \tag{3.18}$$

of which he gave the solutions $R = 31{\cdot}29$ and $R = 237{\cdot}6$. The former number had been given as the critical Rayleigh number governing stability by Yih (1959) (later corrected 1960) whilst earlier Ostrach (1955) had quoted the latter. However, Wooding (1960) after expanding R in ascending powers of α^2 for small α values found that the leading term of the expansion satisfied either (3.15) or (3.16) according as the disturbance was an antisymmetric or a symmetric one. Corresponding to a modal velocity distribution of the symmetric form

$$V(x,y) = [-3^{\frac{1}{2}}\alpha(1-x^2)]\cos\alpha y, \tag{3.19}$$

he obtained a critical Rayleigh number which was zero to a second order in α. This instability which occurs at the limit of zero wave-number is anomalous to the general trend of results for finite wave-numbers as given by Yih (1959) or, in the present instance, by table 1 which shows the more critical mode for a given wave-number to be an antisymmetric one. A physical explanation of this behaviour has been offered by Wooding and is based on the very small variation, on account of the factor α, of the velocity profile (3.19).

The critical values of the Rayleigh number found from (3.13) and (3.14) for stated wave-numbers are displayed in table 1.

The Rayleigh numbers shown in table 1 are 'more critical' than the results for wave-numbers in the vertical direction obtained by Yih (1959). This is increasingly the case for the larger wave-numbers. The corresponding curves for neutral stability, together with Yih's for comparison, are plotted in figures 1 and 2.

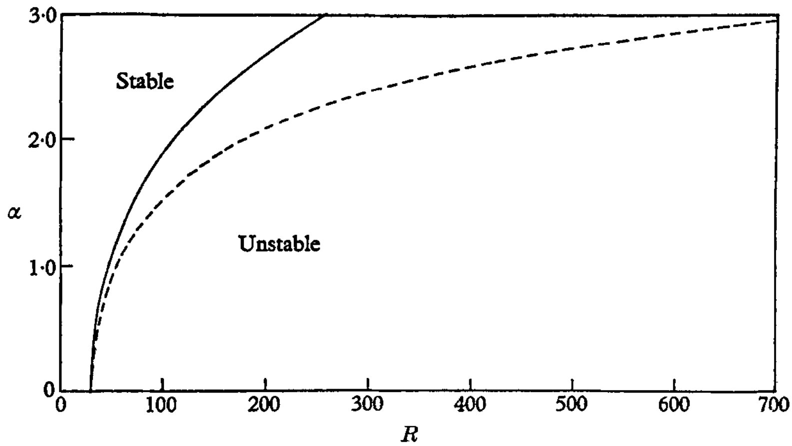

FIGURE 1. Neutral stability curve for antisymmetric convection ($M = 0$). Yih's curve is shown as broken line.

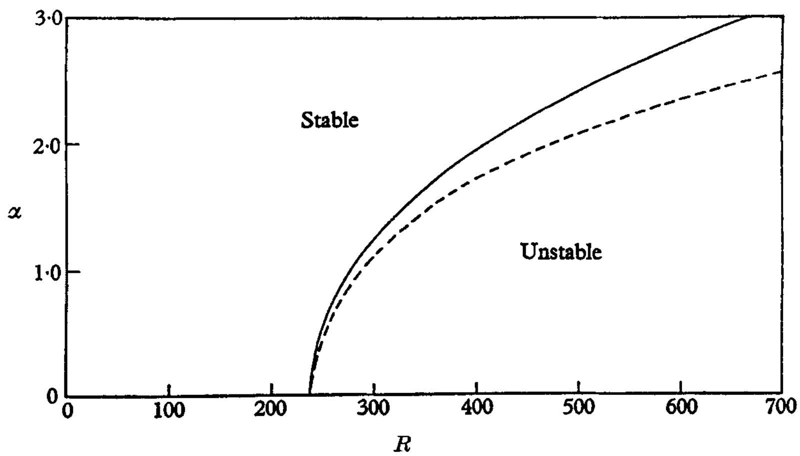

FIGURE 2. Neutral stability curve for symmetric convection ($M = 0$). Yih's curve is shown as broken line.

Case (ii), $M \neq 0$

In this case the critical Rayleigh number appropriate to a mode of convection may be evaluated upon setting $\alpha = 0$ in the secular equations (3.11) and (3.12). These then reduce to the respective forms

$$\xi^{\frac{1}{2}} \tanh(\tfrac{1}{2}\xi)^{\frac{1}{2}} + \zeta^{\frac{1}{2}} \tan(\tfrac{1}{2}\zeta)^{\frac{1}{2}} = 0, \tag{3.20}$$

and

$$\xi^{-\frac{1}{2}} \tanh(\tfrac{1}{2}\xi)^{\frac{1}{2}} - \zeta^{-\frac{1}{2}} \tan(\tfrac{1}{2}\zeta)^{\frac{1}{2}} = 0, \tag{3.21}$$

in which $\xi = (M^4 + 4R)^{\frac{1}{2}} + M^2, \quad \zeta = (M^4 + 4R)^{\frac{1}{2}} - M^2.$

The solutions of (3.20) and (3.21) for varying M are given in table 2, and the neutral stability curves for antisymmetric and symmetric convections are shown in figures 3 and 4.

Evidently $R = 0$ is not a possible solution of either equation in case $M \neq 0$ and it can be concluded that there is no buoyancy force of the previous small

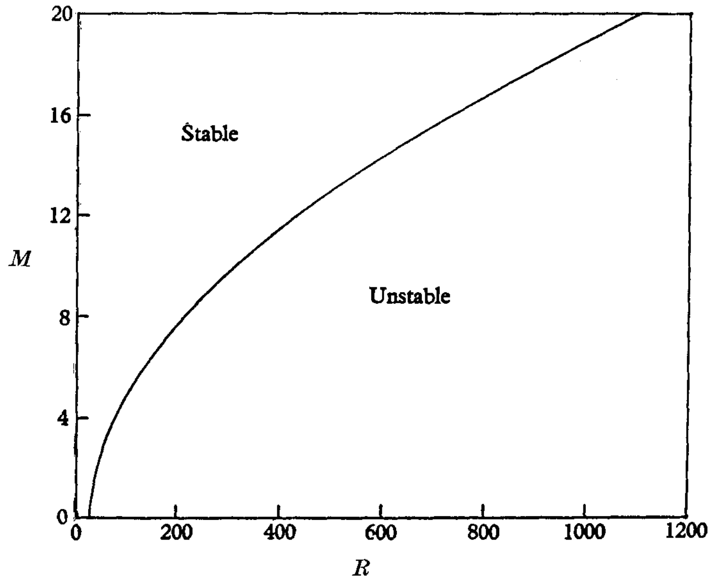

FIGURE 3. Neutral stability curve for antisymmetric convection of zero wave-number.

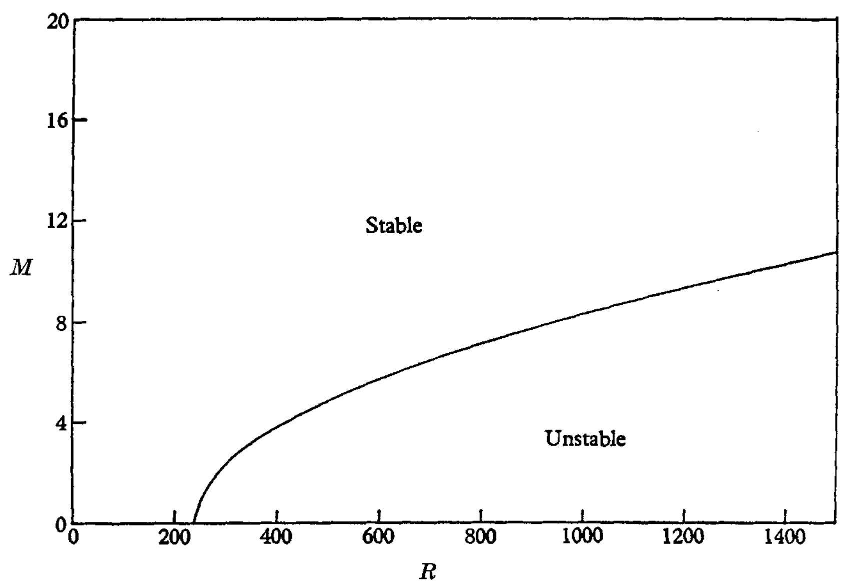

FIGURE 4. Neutral stability curve for symmetric convection of zero wave-number.

scale which could excite a convection current in face of the stabilizing effect of combined viscous and magnetic stresses. Further the results in table 2 indicate that the fluid is more stable to symmetric than to antisymmetric disturbances.

	R	
M	Antisymmetric convection	Symmetric convection
0	31·285	0
2	43·476	283·62
4	79·037	419·73
6	136·27	642·71
10	312·16	1338·3
20	1100·4	4483·3
∞	$(\frac{1}{2}\pi M)^2$	$(\pi M)^2$

TABLE 2. Values of R for given $M(\alpha=0)$.

4. The principle of exchange of stabilities

It has been assumed so far that instability sets in as a steady cellular convection from rest and the possibility of instability (overstability) growing out of existing oscillations has been excluded. It should therefore be shown that the imaginary part of λ in the separated solutions (2.22) is zero at neutral stability.

If the equation (2.23) is operated upon by $(Pr\lambda-\nabla^2)(Pr\lambda-\eta\nabla^2)$ it is possible on using (2.24) and (2.25) to eliminate θ and H and thus obtain the equation

$$(Pr\lambda-\nabla^2)(Pr\lambda-\eta\nabla^2)(\lambda-\nabla^2)V = (Pr\lambda-\eta\nabla^2)RV+\eta M^2(Pr\lambda-\nabla^2)\frac{\partial^2 V}{\partial x^2}. \tag{4.1}$$

The character of (4.1) which is linear and homogeneous in V makes it permissible to consider in general a modal solution of the appropriate form

$$V(x,y) = \cosh \epsilon x \cos \alpha y, \tag{4.2}$$

in which ϵ may be either real or pure imaginary. The substitution of (4.2) into (4.1) then yields

$$[Pr\lambda-(\epsilon^2-\alpha^2)][Pr\lambda-\eta(\epsilon^2-\alpha^2)][\lambda-(\epsilon^3-\alpha^2)] = [Pr\lambda-\eta(\epsilon^2-\alpha^2)]R + \eta M^2[Pr\lambda-(\epsilon^2-\alpha^2)]\omega^2,$$

which can be multiplied out and rearranged in powers of λ to become

$$Pr^2\lambda^3-\lambda^2[(\epsilon^2-\alpha^2)Pr\{1+\eta+Pr\}]+\lambda[(\epsilon^2-\alpha^2)^2\{Pr+\eta Pr+\eta\}-PrR-\eta M^2Pr\epsilon^2] - [\eta(\epsilon^2-\alpha^2)\{(\epsilon^2-\alpha^2)^2-R-M^2\epsilon^2\}] = 0. \tag{4.3}$$

If now λ in (4.3) is replaced by $i\mu$ and the real and imaginary parts of the resulting equation are set equal to zero one has

$$-Pr^2\mu^2+[(\epsilon^2-\alpha^2)^2\{Pr+\eta Pr+\eta\}-PrR-\eta M^2Pr\epsilon^2] = 0, \tag{4.4}$$

and

$$\mu^2[Pr\{1+\eta+Pr\}]-[\eta\{(\epsilon^2-\alpha^2)^2-R-M^2\epsilon^2\}] = 0. \tag{4.5}$$

A single equation for μ^2 can be obtained from (4.4) and (4.5) by eliminating the factor $(\epsilon^2-\alpha^2)^2$ which is common to both. One has from (4.5) that

$$(\epsilon^2-\alpha^2)^2 = \frac{Pr}{\eta}\mu^2[1+\eta+Pr]+R+M^2\epsilon^2.$$

Substituting into (4.4) using this expression for $(\epsilon^2-\alpha^2)^2$ then gives

$$\mu^2[(1+\eta^{-1})Pr\{Pr^2+(1+\eta)Pr+\eta\}]+\eta R(1+Pr)+\epsilon^2M^2(Pr+\eta) = 0. \quad (4.6)$$

Two cases require individual attention according as ϵ is real or imaginary. If $\epsilon^2 > 0$ it is clear from (4.6) that μ cannot be real unless possibly R were negative in which case the temperature gradient β would increase upwards and the fluid would evidently be stable. In case $\epsilon^2 < 0$ the terms independent of μ^2 in (4.6) can still be shown to be positive for positive R. As a preliminary it is first useful to notice that $\epsilon^2M^2(Pr+\eta)$ with ϵ^2 defined by the second of equations (3.9) is more and more negative with increasing M. It is therefore sufficient to consider only large field strengths in which event it is certainly true (cf. table 2) that $R/M^4 \ll 1$. Thus for large M the expression

$$\epsilon^2 = \tfrac{1}{2}[M^2-M^2(1+4\alpha^2/M^2+4R/M^4)^{\frac{1}{2}}]+\alpha^2$$

can be expanded to give

$$\epsilon^2 = -R/M^2+O(R^2/M^6),$$

which in turn enables one to write

$$M^2\epsilon^2(\eta+Pr)+\eta R(1+Pr) = PrR(\eta-1)+O(R^2/M^4). \quad (4.7)$$

From (4.7) it follows that μ cannot be real if $\eta > 1$ since then the terms independent of μ^2 in (4.6) have a positive sum. The more restrictive condition necessary to ensure a real μ arises therefore in connexion with $\epsilon^2 < 0$ and in consequence the validity of the principle of exchange of stabilities is assured where $\eta > 1$. This condition is not a restrictive one on a laboratory scale.

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